VISCOUS FLOW IN A TUBE WITH PERIODIC VARIATION OF TRANSVERSE CROSS SECTION AREA AND DISTRIBUTED INJECTION THROUGH THE LATERAL SURFACE

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Results are provided of a difference scheme simulating the flow of a viscous incompressible fluid in a tube with penetrable walls and discs located inside the tube.

Periodic flows in tubes and channels are encountered in many technological systems and energy devices. As an example we note the technological operation of obtaining dielectric films in a reactor, widely used in semiconducting devices and integrated circuits. One of the first studies, where flow of this type is considered, is, obviously, [1]. Given there is a generalized theory of evolving periodic flow and heat exchange in planar channels. A similar approach was used in [2] to calculate flow and heat exchange in a tube with periodically converging-diverging generating surfaces. A calculation was carried out in [3] of periodic flow in a tube with an infinite sequence of moving cylindrical containers, where to simplify the formulation of periodicity conditions equations were used in variables of the stream function – the vorticity. The problem stated was treated in [4], and results were provided of calculating axially symmetric turbulent flow near an infinite sequence of discs, established in the tube perpendicularly to its axis.

All studies mentioned are characterized by periodic variation of the flow transverse cross section area S:

$$S(x) = S(x+L) = S(x+2L) = \ldots \tag{1}$$

(x is the current coordinate along the tube or channel axis), generating periodic variation of the hydrodynamic parameters. For planar or axially symmetric flow the corresponding relations are

$$u(x,\ y) = u(x+L,\ y) = u(x+2L,\ y) = \ldots; \tag{2}$$

$$v(x,\ y) = v(x+L,\ y) = v(x+2L,\ y) = \ldots; \tag{3}$$

$$p(x,\ y) - p(x+L,\ y) = p(x+L,\ y) - p(x+2L,\ y) = \ldots, \tag{4}$$

where y is the transverse (radial) coordinate.

The last relation makes it possible to represent the pressure variation along the flow in the form of a sum of two components:

$$p(x,\ y) = -\beta x + P(x,\ y), \tag{5}$$

where the first term determines the linear law of pressure drop with gradient β, being uniquely related to fluid discharge into the tube Q, while the second term characterizes the local pressure variation, generated by the local flow deformation. Clearly, since local pressure variations are related to the change in flow geometry, the function P(x, y) is periodic, i.e.,

$$P(x,\ y) = P(x+L,\ y) = P(x+2L,\ y) = \ldots. \tag{6}$$

These periodic flows are investigated in the present paper as applied to the case of flow evolution of a viscous incompressible fluid in a tube of circular transverse cross section with an infinite sequence of discs established inside the tube. Unlike the studies mentioned above, with the purpose of intensifying transport processes in the tube we realize here fluid transfer through the lateral surface of the tube, with the total mass flow

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in the tube varying continuously along its length. We show how this additional condition is reflected in the nature of behavior of the parameters (2)-(6).

Let v_0 be the velocity of uniform fluid delivery through the lateral surface of a tube of radius R. With increasing coordinate x the fluid discharge into the tube Q(x) increases linearly:

$$Q(x) = Q_0 + 2\pi R x v_0, \tag{7}$$

where Q_0 is the discharge through some initial cross section with coordinate x.

Introduce a dimensionless velocity in the longitudinal direction $\hat{u}(x, y)$, defined as the difference between the local velocity u(x, y) and the mean-mass excess velocity $V(x) = 2\pi R x v_0/S(x)$ in functions of the mean-mass velocity u_0 in the selected inlet cross section:

$$\hat{u}(x, y) = [u(x, y) - V(x)]/u_0. \tag{8}$$

The concept of flow evolution in a tube with periodic variation of the transverse cross section area is a logical generalization of the theory of evolution of periodic flow in a tube with an impenetrable wall. For the flow considered it can be assumed that in cross sections removed from each other by distance L the longitudinal velocity profiles u(x, y) are similar to each other. This condition, written for the dimensionless velocity $\hat{u}(x, y)$, is

$$\hat{u}(x, y) = \hat{u}(x + L, y) = \hat{u}(x + 2L, y) = \ldots \tag{9}$$

and is a generalization of condition (2) to the case of periodic flow in a tube with fluid delivery into it through the lateral surface.

We investigate the nature of pressure variation along the length of the tube. For this we consider one-dimensional flow in a tube of constant transverse cross section with given discharge Q_0 at the inlet in the presence of a bulk source fluid uniformly distributed over the whole flow. In this case the fluid discharge over the length of the tube varies linearly: $Q(x) = Q_0(1 + \gamma x)$, where γ is the specific bulk fluid delivery transmitted through unit length and referring to discharge over the input cross section. For the given one-dimensional flow we write the equations of momentum variation and continuity

$$u\frac{\partial u}{\partial x} = -\frac{\partial \overline{P}}{\partial x} + \frac{1}{y}\frac{\partial y\tau_{xy}}{\partial y}; \quad \int_S u\,dS = Q_0(1 + \gamma x), \tag{10}$$

where $\tau_{xy} = \Gamma\partial u/dy$; Γ is the transport coefficient, and $\overline{p}$ is the mean pressure over the cross section.

Integrating the first equation of system (10) over the area of the transverse cross section of the tube, we obtain

$$\frac{\partial}{\partial x}\int_S (u^2/2 + \overline{p})\,dS = -2\pi R\tau_w,$$

where τ_w is the friction at the wall.

We introduce the mean flow velocity over the cross section $\int_S u\,dS = \bar{u}S$ and express the integral of the square of velocity in terms of the mean velocity, as is done in hydraulics:

$$\int_S u^2\,dS = \alpha\bar{u}^2 S - \alpha Q_0^2(1 + \gamma x)^2/(\pi R^2),$$

where α = const is a coefficient, taking into account the nonuniformity of velocity distribution over the cross section (the quantity α is independent of x, since for uniform fluid delivery from the distributed internal source, obviously, is satisfied by the similar velocity profile).

We rewrite the equation of variation of motion momentum in the form

$$\frac{d\overline{p}}{dx} = -\frac{d}{dx}\left[\alpha\frac{u_0^2}{2}(1 + \gamma x)^2\right] - \frac{2\tau_w}{R}.$$

Unlike uniform flow in a tube, for which τ_w = const, in the flow considered with similar longitudinal velocity it can be assumed that friction at the wall varies by the law $\tau_w = \tau_{w0}(1 + \gamma x)$, where τ_{w0} is the friction at the wall at the input cross section of the computational region. To sum up, we obtain that the pressure gradient along the length of the tube decreases linearly

$$\frac{d\overline{p}}{dx} = -\beta(1+\gamma x),$$

where $\beta = \alpha u_0^2\gamma + 2\tau_{w0}/R$.

Hence follows the quadratic law of pressure variation in a tube with a distributed internal source:

$$\overline{p} = \overline{p}_0 - \beta x - \beta\gamma x^2/2.$$

Turning to periodic flow in a tube with uniform fluid delivery through the lateral surface, it is logically assumed that the nature of pressure variation due to the total mass flow in the tube is the same as for the model problem considered above. Consequently, the pressure field can be represented as:

$$p(x,\ y) = -\beta x - \beta\gamma x^2/2 + P(x,\ y), \tag{11}$$

where the function $P(x, y)$, related to the local flow character, is periodic and obeys conditions (6).

We turn now to investigate a specific flow. The computational module has length L, and is included between two neighboring transverse cross sections with a disc established in the middle. The transverse cross section area varies by the law $S(x) = \pi R^2$ for $x \neq L/2$ and $S(x) = \pi R^2(1-\theta)^2$ for $x = L/2$, where θ is the ratio of the disc radius to the tube radius.

The system of equations describing the flow is

$$\begin{aligned}
&\frac{\partial \hat{u}}{\partial x} + \frac{1}{y}\frac{\partial yv}{dy} = -\frac{dV}{dx};\quad \frac{\partial y\hat{u}^2}{\partial x} + \frac{\partial y\hat{u}v}{\partial y} = \beta(1+\gamma x) - \frac{\partial P}{\partial x} + \\
&+\frac{\partial}{\partial x}\left(y\Gamma\frac{\partial \hat{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\Gamma\frac{\partial \hat{u}}{\partial y}\right) - 2y\hat{u}\frac{dV}{dx} + V\frac{\partial yv}{\partial y} + \\
&+\frac{\partial}{\partial x}\left(\Gamma\frac{\partial V}{\partial x}\right);\quad \frac{\partial y\hat{u}v}{\partial x} + \frac{\partial yv^2}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(y\Gamma\frac{\partial v}{\partial x}\right) + \\
&+\frac{\partial}{\partial y}\left(y\Gamma\frac{\partial v}{\partial y}\right) - yv\frac{dV}{\partial x} - yV\frac{\partial v}{\partial x}.
\end{aligned} \tag{12}$$

Here the velocity components refer to the mean mass velocity u_0 in the inlet cross section of the computational module, while the pressure and transport coefficient are normalized, respectively, by ρu_0^2 and u_0R. The velocity V is determined with account of the relations given above, and is normalized by u_0.

The system of equations (12) is solved for the periodicity conditions (3), (6), (9), and for the following boundary conditions: on the $y = 0$ axis $v = 0$, $\partial\hat{u}/\partial y = 0$; on the tube wall $y = 1$, $v = v_0u_0$, $\hat{u} = -V(x)$; and on the disc surfaces at $x = L/2$, $y \leq \theta$, $v = 0$, $\hat{u} = -V(L/2)$. By its definition the coefficient γ equals $\gamma = 2\pi v_0RL/(u_0\pi R^2L) = 2v_0/u_0$. To determine the coefficient β we use the continuity equation in integral form

$$\int_S (1-u)\,dS = 0. \tag{13}$$

It is noted that Eq. (13) is properly used only in one flow cross section, primarily at the outlet. In this case $V(x) = 0$, the dimensionless velocity u coincides with $\hat{u}$, and the integral in Eq. (13) can be calculated from the integration results of the system of equations (12).

To solve the system of equations (12) we use the control volume method according to the simple method or one of its modifications (see, for example, the studies [5, 6]), taking into account the variations generated by the necessity of the determining the coefficient by means of Eq. (13). Consider the features of the solution method used here. Application of the control volume method to the equation of momentum variation (12) leads to algebraic forms of the shape

$$a_P\Phi_P = a_E^{\Phi}\Phi_E + a_W^{\Phi}\Phi_W + a_N^{\Phi}\Phi + a_S^{\Phi}\Phi_S + S_{\Phi},$$

where $\Phi = \hat{u}$, v, in which the coefficients and source term are calculated as a function of the approximation scheme adopted of convective and diffusion terms velocity values $\hat{u}$, v, at the given iteration, periodic pressure P components at the grid sites, and coefficient β.

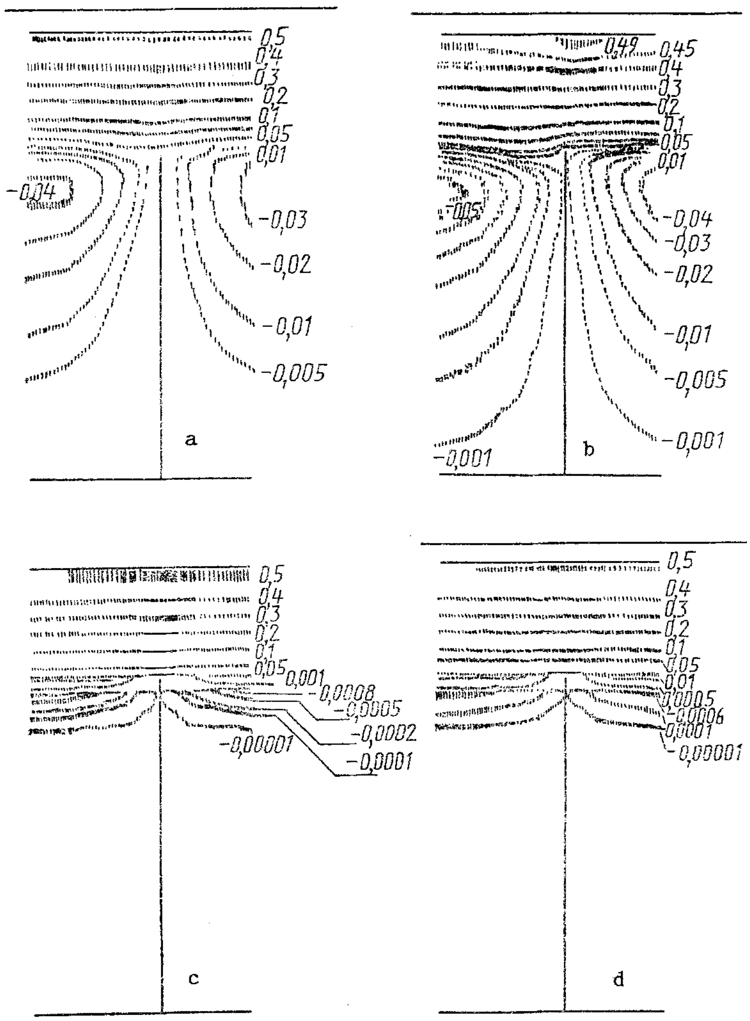

Fig. 1. The stream function field in the computational region in the case of impenetrable tube walls for various values of the Reynolds number and distances between discs with radius $\theta = 0.75$; a) $L = 0.6$; $Re = 500$; b) 0.6 and 50; c) 0.1 and 50; d) 0.1 and 5.

The values found of $\hat{u}^*$ and v^* are assumed to be preliminary, and are later corrected in such a manner that they satisfy the continuity equations of the system (12) and (13). For this we introduce the correcting dependences $\hat{u} = \hat{u}^* + \hat{u}'\ \hat{u}''$, $v = v^* + v'$, $P = P^* + P'$, $\beta = \beta^* + \beta'$.

Unlike the standard SIMPLE procedure, the velocity component $\hat{u}$ has two corrections ($\hat{u}'$ and $\hat{u}''$), where the corrections $\hat{u}'$ and v' are expressed in terms of the corresponding difference of corrections to the pressure function P', while the second velocity correction $\hat{u}''$ is calculated in terms of the correction to the linear pressure gradient component β'. The corrective dependence for the pressure function P' is obtained, as in the SIMPLE procedure, in the form of a difference Poisson equation, while the correction β' is found by the equation [4, 6]:

$$\beta' = \sum [(1 - u_P^*)\,\Delta A] \Big/ \sum \left(\frac{\mathrm{vol}_P}{a_P^u} \Delta A \right),$$

in which the summation is carried over all sites located in the inlet cross section. Here vol_P is the value of the control volume, and ΔA is the boundary area in the transverse direction.

We discuss the results of calculations obtained on the basis of the theory and method developed. We first dwell on the features of flow periodicity in the case of a tube with impenetrable walls. Figure 1 shows the stream function field within the computational re-

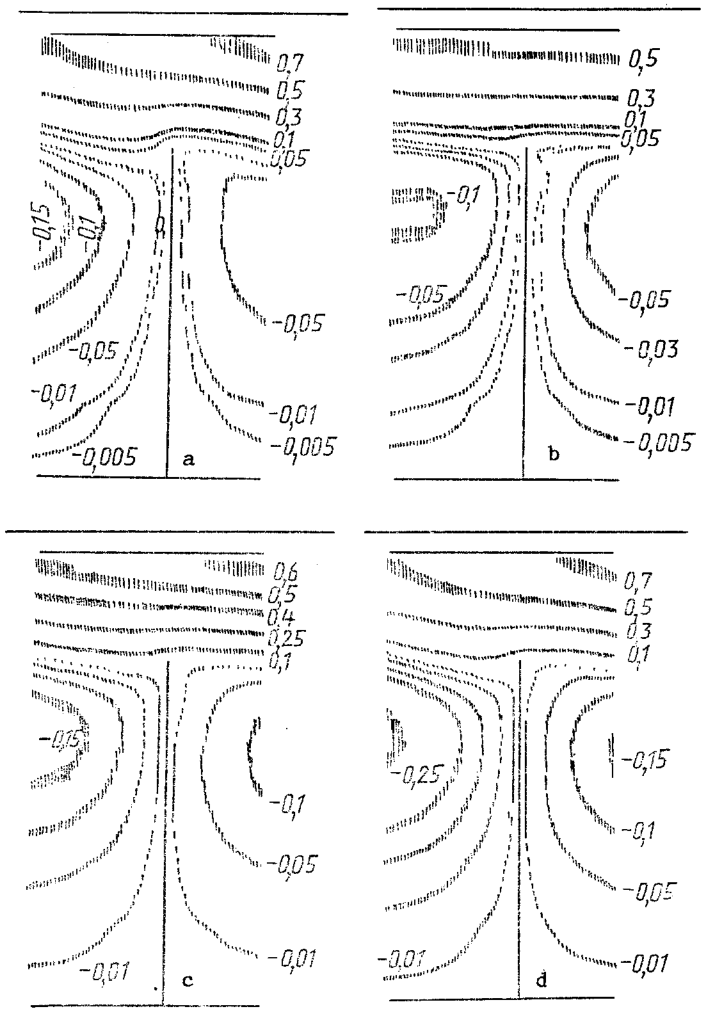

Fig. 2. The stream function field during fluid delivery through the tube wall in a module of size L = 0.6 with disc radius $\theta = 0.75$ for various values of the Reynolds number and fluid delivery rate: a) Re = 50; $v_0 = 0.5$; b) 500 and 0.1; c) 500 and 0.25; d) 500 and 0.5.

gion for various values of the Reynolds number Re and the module size L. For relatively small L values ($L < 0.8$) the flow can be divided conditionally into two parts: the flow in annular space, confined by the tube surface and the coaxial cylindrical surface covering the disc roots, and the flow in the cylindrical space between the discs. The stream line pattern shows that in the annular region mentioned the flow approximates flow in a circular tube. The stream lines are almost rectilinear, their deformation is observed only near the disc edges, and in several cases, due to local breaks of low intensity – at the tube surface, while the deformation indicated is enhanced with decreasing Reynolds number Re and increasing interdisc distance L. Due to the flow discontinuity at the edges, vortical fluid motion with formation of a large-scale vortex is generated in the space between the discs. The longitudinal vortex size equals to the distance L in all regimes, while the transverse size is primarily different from the size L. With decreasing L the radial size of the vortex also decreases, it tightens down to the peripheral part of the computational region, and a stagnant region is formed directly near the axis. An increase in the Reynolds number leads to enhancement in the vortex formation pattern, as well as to attenuation of the relative intensity of the return flow formed. The intensity of the return flow drops sharply with decreasing distance L, in which case the effect of the Reynolds number on the nature of formation of return flow is reduced.

Fluid delivery through the lateral tube surface varies substantially the flow pattern in the annular space over the disc edges (Fig. 2). Near the tube wall the stream lines have a slope, which is higher the larger the relative velocity of fluid delivery v_0. However,

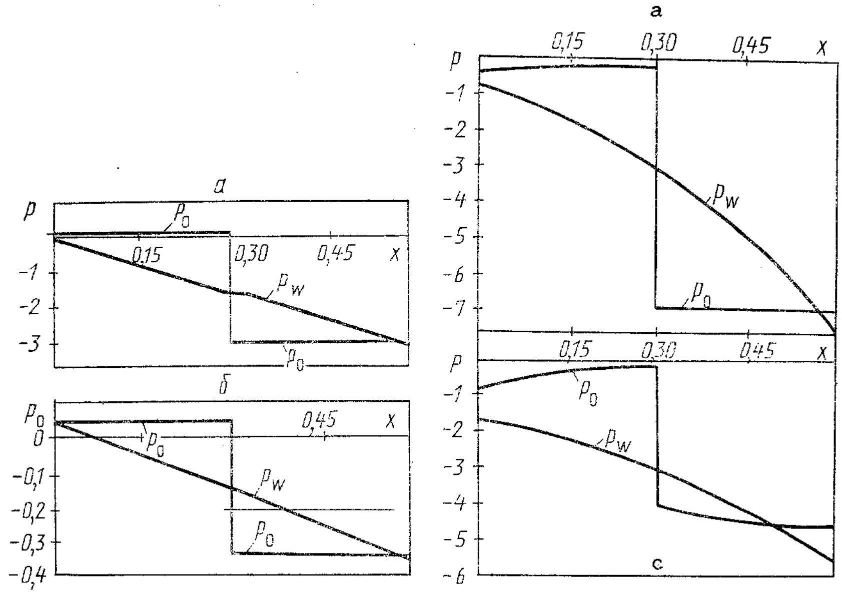

Fig. 3. Pressure variation over the module length for L = 0.6 and θ = 0.75 at the tube surface and axis in a tube with an impenetrable wall (a, b) and for fluid delivery through a tube wall with relative velocity $\bar{v}_0$ = 0.25 (c, d) for various Reynods number values: a, b; Re = 50; c, d; 500.

in the fluid layer directly adjacent to disc edges the nature of low is close to that which would occur in a tube with an impenetrable wall. The stream lines approach here the straight line parallel to the axis, though their extent of deformation is higher (see Figs. 1, 2). In this case one large-scale vortex is formed here, whose intensity increases with increasing fluid delivery rate v_0. For low L values vortex generation in the opposite direction is possible in the region adjacent to the leeward side of the disc and to the axis. The intensity of this opposite vortex is insignificant, it narrows down the basic contribution of the main vortex to the windward side of the disc below the flow and to the annular flow region, and can even divide it into two vortices of lesser size and intensity.

Not dwelling on the features of behavior of other parameters, we turn to the study of pressure variation over the flow and the hydraulic resistance of the tube-disc system under consideration. Figure 3 shows plots for the pressure at the tube surface p_w and at the axis p_0 for values of the Reynolds number Re = 50 and Re = 100 in the cases of an impenetrable wall and fluid delivery through the tube wall with relative velocity $\bar{v}_0$ = 0.25. The plots constructed reflect the fact noted above of existence in the flow of two different flow types — the tube in the annular gap between the disc edges, and the tube with stagnant vortex in the space between discs.

For an impenetrable wall the pressure at the tube surface p_w decreases almost linearly, except for a small region near the disc, coinciding with the nature of pressure behavior in the case of evolution of laminar fluid flow in a tube. Fluid delivery through the lateral surface leads to a nearly parabolic pressure variation law with length p_w, corresponding to the theoretical discussion presented above. The behavior of the p_w curves shows that the sharp disc edges at relatively small L values (L < 0.6) deform slightly the annular flow, carrying the basic mass of the fluid.

In the case of an impenetrable wall, on the axis and in the space between neighboring discs the pressure p_0 is conserved, reflecting the existence of a stagnant zone or a zone of weak vortical flow between the discs. For fluid delivery through the lateral surface vortical motion is substantially activated, particularly for large Reynolds number values. As a result the pressure at the axis becomes nonuniform — upon moving away from the windward side of the disk the quantity p_0 initially drops, and then increases with direction toward the windward side of the disc from below the flow. The pressure varies jumpwise upon transition through the disc location cross section.

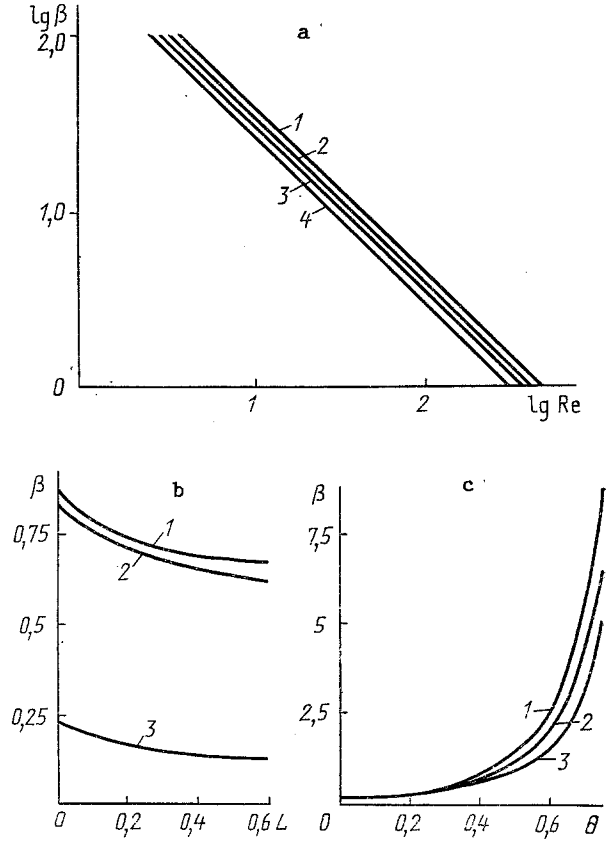

Fig. 4. The dimensionless pressure gradient β, averaged over the module length, as a function of the Reynolds number (a), the module length (b), and the disc radius (c) (L = 0 – Eq. (14); L = 0.1, 0.2, and 0.6 – calculated for Re = 5, 50, 500 and θ = 0.75 (a, b)): a) 1 – theory (L = 0), 2-4 – calculation (L = 0.1, 0.2, 0.6); a) 1 – β (Re = 500); 2 – β/10 (Re = 50); 3 – β/100 (Re = 5); c) 1 – L = 0; 2 – 0.1; 3 – 0.6 (Re = 50).

The hydraulic resistance of the tube-disc system under consideration is characterized by the dimensionless parameter β, which for uniform flow in a tube with impenetrable walls is normalized over the mean-discharge flow rate of pressure gradient, and for fluid delivery through the lateral surface – by the linear part in the parabolic pressure variation law. Since the pressure drop is basically determined by the flow in the annular region between the tube surface and the disc edges, it is logical to carry out a comparison with the similar quantity for flow in a circular tube of width 1 – θ. Using, to eliminate dimensions, the characteristic values selected earlier, more precisely the tube radius R and the mean flow velocity over the tube cross section u_0, the equation for determining the dimensionless pressure gradient in a circular tube is represented in the form

$$\beta_0 = \frac{12}{\mathrm{Re}} \frac{1}{(1-\theta)^3 (1+\theta)}. \tag{14}$$

For a circular tube with fluid transfer through the lateral surface, to determine β we integrate Eq. (10) over the segment $1 - \theta < y/R < 1$. Since the flow obeys in the present case the same laws as flow in a circular tube with a similar penetrable wall, on the basis of the discussion above one can write

$$\frac{dp^*}{dx} = \frac{1}{y} \frac{\partial}{\partial y} \left(y\Gamma \frac{\partial u^*}{\partial y} \right),$$

where dp^*/dx denotes the quantity $[1/(1+\gamma x)]\, d/dx\, [P + \alpha U^2 (1+\gamma x)^2/2]$; $u^* = u(1+\gamma x)$; $\gamma = 2v_0/[UR(1-\theta^2)]$; $U = Q/[\pi R^2 (1-\theta^2)]$ is the mean-mass velocity in the inlet cross section of the circular tube, and Q is the fluid discharge in the inlet cross section.

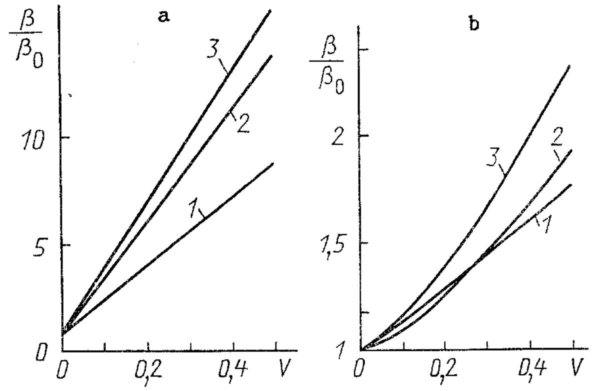

Fig. 5. The relative hydraulic resistance coefficient β/β_0 as a function of the relative fluid delivery rate through the lateral surface of the tube for various distances between discs and various values of the Reynolds number Re = 500 (a) and 50 (b): 1) L = 0 (theory, Eq. (15)); 2, 3) L = 0.1 and 0.6 (calculated for θ = 0.75).

The equation given is similar to the equation describing flow in a circular tube with impenetrable walls. Using for the characteristic velocity the quantity $u_0 = Q/(\pi R^2)$, so that $U = u_0/(1 - \theta^2)$, relating p^* to ρu_0^2 and x to R, and taking into account that at the inlet cross section $u = u^*$ for $x = 0$, we obtain for the dimensionless gradient $\beta^* = (dp^*/dx)R/u_0^2$ the same value as in Eq. (14). Turning from the function p^* to the pressure p, denoting the dimensionless pressure gradient at $x = 0$ by $\beta_0[(\partial p/\partial x)R/\rho u_0^2]_0$, and taking into account that the coefficient α for circular laminar flow equals 1.5, we find a dependence for β of the form

$$\beta = \beta_0 + \frac{3}{(1-\theta^2)^2} V, \qquad (15)$$

where β_0 is determined by Eq. (14), and $V = v_0/u_0$.

Figure 4 shows plots of the dimensionless pressure gradient in a tube with an impenetrable wall as a function of the Reynolds number and of the geometric parameters L and θ. The value L = 0 corresponds to the case of flow in a circular tube. It is seen from Fig. 4a that the flow periodicity due to disc location in the tube does not change the inversely proportional dependence of the coefficient β on the Reynolds number. The proportionality coefficient in the dependence β(Re) decreases with increasing L. This implies that a continuous internal wall of a circular tube generates in the flow a larger pressure loss than a periodic sequence of bulging discs and a stagnant zone between them, confining the circular flow inside. The given conclusion deserves attention independently of the practical consideration provided above for these flows.

A representation of the quantitative variation of β with increasing L is given by Fig. 4b. It is seen that with increasing Reynolds number the drop rate in the quantity β increases with increasing periodicity parameter L. The nature of β variation with increasing disc radius (the parameter θ) is determined by the curves in Fig. 4c. With increasing θ the enhancement in β occurs faster than the distance between discs decreases.

The effect of fluid delivery on the hydraulic resistance of the tube-disc system is characterized by the plots showing the variation in the coefficient β/β_0 (Fig. 5). With increasing fluid delivery rate through the wall of the tube (the parameter V) the coefficient β increases nearly linearly. It is noted that the theoretical curve 1 was constructed by Eq. (15) and determines, as already mentioned, the resistance of a circular tube. The plots show that for fluid delivery through the exterior surface of the tube the hydraulic resistance of the tube-disc system increases, becoming substantially larger than for the same fluid delivery in a circular tube. This effect is manifested more substantially for high values of the Reynolds number. As to the effect on the resistance of geometric sizes, and as is seen from the results given, the hydraulic resistance of the tube-disc system increases with increasing distance between discs.

In conclusion we note one more interesting and practically important feature of the results, at least for the class of problems under consideration. The essence is that for small values of the Reynolds number (Re = 50) the hydraulic resistance of the tube-disc system with fluid delivery through the lateral surface of the tube is below the corresponding value for a circular tube (L = 0) for small values of the fluid delivery rate (V < 0.3).

NOTATION

Here R denotes the tube radius, θ is the relative radius of the disc, L is the distance between discs (module size), u, v are the velocity projections on the coordinate axes, Q_x is the discharge in an arbitrary cross section, u_0 is the mean-mass velocity at the inlet cross section, V is the dimensionless injection rate, $\hat{u}$ is the excess dimensionless velocity in the axial direction, p is the pressure, P is the pressure component related to local vortical motion, β is the linear pressure component due to the total mass flow, γ is the quadratic pressure component due to injection through the lateral surface, and Re denotes the Reynolds number.

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